

REGULARITY OF RADIAL EXTREMAL SOLUTIONS FOR SOME NON LOCAL SEMILINEAR EQUATIONS

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ABSTRACT. We investigate stable solutions of elliptic equations of the type

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } B_1 \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $n \geq 2$, $s \in (0, 1)$, $\lambda \geq 0$ and f is any smooth positive superlinear function. The operator $(-\Delta)^s$ stands for the fractional Laplacian, a pseudo-differential operator of order $2s$. According to the value of λ , we study the existence and regularity of weak solutions u .

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1. INTRODUCTION

We are interested in the regularity properties of stable solutions satisfying the following semilinear problem involving the fractional Laplacian

$$(1.1) \quad \begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Here, B_1 denotes the unit-ball in \mathbb{R}^n , $n \geq 2$, and $s \in (0, 1)$. The operator $(-\Delta)^s$ is defined as follows: let $\{\varphi_k\}_{k=1}^\infty$ denote an orthonormal basis of $L^2(B_1)$ consisting of eigenfunctions of $-\Delta$ in B_1 with homogeneous Dirichlet boundary conditions, associated to the eigenvalues $\{\mu_k\}_{k=1}^\infty$. Namely, $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_k \rightarrow +\infty$, $\int_{B_1} \varphi_j \varphi_k \, dx = \delta_{j,k}$ and

$$\begin{cases} -\Delta \varphi_k = \mu_k \varphi_k & \text{in } B_1 \\ \varphi_k = 0 & \text{on } \partial B_1. \end{cases}$$

The operator $(-\Delta)^s$ is defined for any $u \in C_c^\infty(B_1)$ by

$$(-\Delta)^s u = \sum_{k=1}^\infty \mu_k^s u_k \varphi_k,$$

where

$$u = \sum_{k=1}^\infty u_k \varphi_k, \quad \text{and} \quad u_k = \int_{B_1} u \varphi_k \, dx.$$

This operator can be extended by density for u in the Hilbert space

$$(1.2) \quad H = \{u \in L^2(B_1) : \|u\|_H^2 = \sum_{k=1}^\infty \mu_k^s |u_k|^2 < +\infty\}.$$

Note that

$$H = \begin{cases} H^s(B_1) & \text{if } s \in (0, 1/2), \\ H_{00}^{1/2}(B_1) & \text{if } s = 1/2, \\ H_0^s(B_1) & \text{if } s \in (1/2, 1), \end{cases}$$

see Section 2 for further details. In all cases, $(-\Delta)^s : H \rightarrow H'$ is an isometric isomorphism from H to its topological dual H' . We denote by $(-\Delta)^{-s}$ its inverse, i.e. for $\psi \in H'$, $\varphi = (-\Delta)^{-s}\psi$ if φ is the unique solution in H of $(-\Delta)^s\varphi = \psi$.

We will assume that the nonlinearity f is smooth, nondecreasing,

$$(1.3) \quad f(0) > 0, \quad \text{and} \quad \lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty.$$

In the spirit of [3], weak solutions for (1.1) are defined as follows: let $\varphi_1 > 0$ denote the eigenfunction associated to the principal eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary condition on B_1 , normalized by $\|\varphi_1\|_{L^2(B_1)} = 1$.

Definition 1.1. *A measurable function u in B_1 such that $\int_{B_1} |u|\varphi_1 dx < +\infty$ and $\int_{B_1} f(u)\varphi_1 dx < +\infty$, is a weak solution of (1.1) if*

$$(1.4) \quad \int_{B_1} u\psi dx = \lambda \int_{B_1} f(u)(-\Delta)^{-s}\psi dx,$$

for all $\psi \in C_c^\infty(B_1)$.

The right-hand side in (1.4) is well defined, since for every $\psi \in C_c^\infty(B_1)$, there exists a constant $C > 0$ such that $|(-\Delta)^{-s}\psi| \leq C\varphi_1$, see Lemma 3.1 and its proof.

We shall be interested in weak solutions of (1.1) having the following stability property.

Definition 1.2. *A weak solution u of (1.1) is semi-stable if for all $\psi \in C_c^\infty(B_1)$ we have*

$$(1.5) \quad \int_{B_1} |(-\Delta)^{\frac{s}{2}}\psi|^2 dx \geq \int_{B_1} f'(u)\psi^2 dx.$$

The following result gives the existence of solutions according to the values of λ .

Proposition 1.3. *Let $s \in (0, 1)$. There exists $\lambda^* > 0$ such that*

- *for $0 < \lambda < \lambda^*$, there exists a minimal solution $u_\lambda \in H \cap L^\infty(B_1)$ of (1.1). In addition, u_λ is semi-stable and increasing with λ .*
- *for $\lambda = \lambda^*$, the function $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a weak solution of (1.1). We call λ^* the extremal value of the parameter and u^* the extremal solution.*
- *for $\lambda > \lambda^*$, (1.1) has no solution $u \in H \cap L^\infty(B_1)$.*

For the proof, see Section 3.

Remark. Proposition 1.3 remains true when B_1 is replaced by any smoothly bounded domain.

Remark. For $0 < \lambda < \lambda^*$, the solution u_λ is minimal in the sense that $u_\lambda \leq u$ for any other weak solution u . In particular, u_λ and u^* are radial. In addition, u_λ and u^* are radially decreasing (see Section 4) and $u_\lambda \in C^\infty(B_1) \cap C^\alpha(\overline{B_1})$ for $\alpha \in (0, \min(2s, 1))$ (see Section 2). If u^* is bounded, then we also have $u^* \in C^\infty(B_1) \cap C^\alpha(\overline{B_1})$ for $\alpha \in (0, \min(2s, 1))$, using again Section 2.

Here is our main result, concerning the regularity of the extremal solution u^* .

Theorem 1.4. Assume $n \geq 2$ and let u^* be the extremal solution of (1.1). We have that:

- (a) If $n < 2(s + 2 + \sqrt{2(s + 1)})$ then $u^* \in L^\infty(B_1)$.
- (b) If $n \geq 2(s + 2 + \sqrt{2(s + 1)})$, then for any $\mu < n/2 - 1 - \sqrt{n - 1} - s$, there exists a constant $C > 0$ such that $u^*(x) \leq C|x|^{-\mu}$ for all $x \in B_1$.

Remark. In particular, for any $2 \leq n \leq 6$, any $s \in (0, 1)$, and any smooth nondecreasing f such that (1.3) holds, the extremal solution is always bounded.

Remark. We do not know if the bound $n < 2(s + 2 + \sqrt{2(s + 1)})$ is optimal for the regularity of u^* . We note however that $\lim_{s \rightarrow 1^-} 2(s + 2 + \sqrt{2(s + 1)}) = 10$, and that the extremal solution of

$$(1.6) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

is singular when $\Omega = B_1$, $f(u) = e^u$, and $n = 10$ (see e.g. [17]).

Nonlinear equations involving fractional powers of the Laplacian are currently actively studied. Caffarelli, Salsa and Silvestre studied free boundary problems for such operators in [10, 11]. Cabré and Tan [8] obtained several results in analogy with the classical Lane-Emden problem $-\Delta u = u^p$, posed on bounded domains and entire space, such as the role of the critical exponent. Previously, some authors considered elliptic equations with nonlinear Neumann boundary condition, which share some properties with semilinear equations of the form (1.1), see e.g. [7, 13].

Equation (1.1) is the fractional Laplacian version of the classical semilinear elliptic equation (1.6). When $f(u) = e^u$, (1.6) is known as the Liouville equation [20] or the Gelfand problem [15]. Joseph and Lundgren [17] showed in this case that if Ω is a ball, then the extremal solution u^* of (1.6) is bounded if and only if $n < 10$. Crandall and Rabinowitz [12] and Mignot and Puel [21] proved that if $f(u) = e^u$ and $n < 10$ then for any smoothly bounded domain Ω , u^* is bounded. Using Hardy's inequality, Brezis and Vázquez [4] provided a different proof that u^* is singular when $\Omega = B_1$ and $n \geq 10$. For some other explicit nonlinearities, such as $f(u) = (1 + u)^p$ with $p > 1$ or $p < 0$, the critical dimension for the regularity of the extremal solution is known (for further details see the above mentioned references). For general nonlinearities, Nedev [22] proved that for any convex function f satisfying (1.3), and any smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, u^* is bounded. This result has been extended by Cabré to the case $n = 4$ and Ω strictly convex [5]. Finally, Cabré and Capella [6] showed that if Ω is a ball and $n \leq 9$ then for any nonlinearity f satisfying (1.3), the extremal solution is bounded.

2. PRELIMINARIES

2.1. Functional spaces. We start by recalling some functional spaces, see for instance [19, 23]. For $s \geq 0$, $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$$

where \hat{u} denotes the Fourier transform of u , with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^s) \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

This norm is equivalent to

$$\|u\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$, the space $H^s(\Omega)$ is defined as the set of functions $u \in L^2(\Omega)$ for which the following norm is finite

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

An equivalent construction consists of restrictions of functions in $H^s(\mathbb{R}^n)$. We define $H_0^s(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. It is well known that for $0 < s \leq \frac{1}{2}$, $H_0^s(\Omega) = H^s(\Omega)$, while for $1/2 < s < 1$ the inclusion $H_0^s(\Omega) \subseteq H^s(\Omega)$ is strict (see Theorem 11.1 in [19]).

The space H defined in (1.2) is the interpolation space $(H_0^2(\Omega), L^2(\Omega))_{s,2}$, see for example [2, 19, 23]. Here we follow the notation from [23, Chap. 22]. J.-L. Lions and E. Magenes [19] showed that $(H_0^2(\Omega), L^2(\Omega))_{s,2} = H_0^s(\Omega)$ for $0 < s < 1$, $s \neq 1/2$, while

$$(H_0^2(\Omega), L^2(\Omega))_{1/2,2} = H_{00}^{1/2}(\Omega)$$

where

$$H_{00}^{1/2}(\Omega) = \{u \in H^{1/2}(\Omega) : \int_{\Omega} \frac{u(x)^2}{d(x)} dx < +\infty\},$$

and $d(x) = \text{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

An important feature of the operator $(-\Delta)^s$ is its nonlocal character, which is best seen by realizing the fractional Laplacian as the boundary operator of a suitable extension in the half-cylinder $\Omega \times (0, \infty)$. Such an interpretation was demonstrated in [10] for the fractional Laplacian in \mathbb{R}^n . Their construction can easily be extended to the case of bounded domains as described below.

Let us define

$$\begin{aligned} \mathcal{C} &= \Omega \times (0, +\infty), \\ \partial_L \mathcal{C} &= \partial\Omega \times [0, +\infty). \end{aligned}$$

We write points in the cylinder using the notation $(x, y) \in \mathcal{C} = \Omega \times (0, +\infty)$.

Given $s \in (0, 1)$, consider the space $H_{0,L}^1(y^{1-2s})$ of measurable functions $v : \mathcal{C} \rightarrow \mathbb{R}$ such that $v \in H^1(\Omega \times (s, t))$ for all $0 < s < t < +\infty$, $v = 0$ on $\partial_L \mathcal{C}$ and for which the following norm is finite

$$\|v\|_{H_{0,L}^1(y^{1-2s})}^2 = \int_{\mathcal{C}} y^{1-2s} |\nabla v|^2 dx dy.$$

Proposition 2.1. *There exists a trace operator from $H_{0,L}^1(y^{1-2s})$ into $H_0^s(\Omega)$. Furthermore, the space H given by (1.2) is characterized by*

$$H = \{u = \text{tr}_{\Omega} v : v \in H_{0,L}^1(y^{1-2s})\}.$$

Proof. For the case $s = 1/2$ see Proposition 2.1 in [8].

We consider now $s \neq 1/2$. Restating the results of Paragraph 5 of J.-L. Lions [18], there exists a constant $C > 0$ such that

$$\|v(\cdot, 0)\|_{H^s(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s} (v^2 + |\nabla v|^2) dx dy,$$

whenever the right-hand side in the above inequality is finite. Now for any $v \in H_{0,L}^1(y^{1-2s})$,

$$\int_{\mathcal{C}} y^{1-2s} v^2 \, dx dy \leq C \int_{\mathcal{C}} y^{1-2s} |\nabla v|^2 \, dx dy,$$

as follows from the standard Poincaré inequality in Ω . Hence, extending v by zero outside \mathcal{C} , we deduce that

$$\|v(\cdot, 0)\|_{H^s(\Omega)} \leq C \|v\|_{H_{0,L}^1(y^{1-2s})}.$$

This inequality shows that there exists a linear bounded trace operator

$$tr_{\Omega} : H_{0,L}^1(y^{1-2s}) \rightarrow H^s(\Omega).$$

This operator has its image contained in $H_0^s(\Omega)$. This is direct for $0 < s < 1/2$ because in this case $H_0^s(\Omega) = H^s(\Omega)$. If $1/2 < s < 1$ we argue that any $v \in H_{0,L}^1(y^{1-2s})$ can be approximated by functions in $H_{0,L}^1(y^{1-2s})$ that have support away from $\partial_L \mathcal{C}$. The trace of any such function has compact support in Ω and is therefore in $H_0^s(\Omega)$. In all cases, this implies that the image of the trace operator is contained in H .

Let us prove $tr_{\Omega} : H_{0,L}^1(y^{1-2s}) \rightarrow H$ is surjective. Take a function $u \in H$ and let us prove that there exists $v \in H_{0,L}^1(y^{1-2s})$ such that $tr_{\Omega}(v) = u$. Write its spectral decomposition $u(x) = \sum_{k=1}^{+\infty} b_k \varphi_k(x)$ and consider the function

$$(2.1) \quad v(x, y) = \sum_{k=1}^{+\infty} b_k \varphi_k(x) g_k(y),$$

where g_k satisfies

$$(2.2) \quad g_k'' + \frac{1-2s}{y} g_k' - \mu_k g_k = 0 \quad \text{in } (0, +\infty)$$

$$(2.3) \quad g_k(0) = 1 \quad g_k(+\infty) = 0.$$

This ODE is a Bessel equation. Two independent solutions are given by $y^s I_s(\sqrt{\mu_k} y)$ and $y^s K_s(\sqrt{\mu_k} y)$, where I_s, K_s are the modified Bessel functions of the first and second kind, see [1]. Since I_s increases exponentially at infinity and K_s decreases exponentially, the solution we are seeking has the form

$$g_k(y) = c_k y^s K_s(\sqrt{\mu_k} y).$$

It is well-known that $K_s(t) = at^{-s} + o(t^{-s})$ as $t \rightarrow 0$, where $a > 0$. Therefore, one can choose c_k such that $g_k(0) = 1$ and one can see that g_k can be written in the form

$$g_k(y) = h(\sqrt{\mu_k} y),$$

for a fixed function h that verifies $h(0) = 1$ and $h'(t) = -ct^{2s-1} + o(t)$ as $t \rightarrow 0$, for some constant $c = c_{n,s} > 0$ depending only on s and n . This implies that

$$(2.4) \quad \lim_{y \rightarrow 0^+} -y^{1-2s} g_k'(y) = c_{n,s} \mu_k^s.$$

Since each of the functions g_k decreases exponentially at infinity we see that v defined by (2.1) is smooth for $y > 0$, $x \in \Omega$ and moreover satisfies

$$\operatorname{div}(y^{1-2s} \nabla v) = 0 \quad \text{in } \mathcal{C}.$$

Let us check that $v \in H_{0,L}^1(y^{1-2s})$. For any $y > 0$, by the properties of φ_k :

$$\int_{\Omega} |\nabla v(x, y)|^2 dx = \sum_{k=1}^{\infty} b_k^2 (\mu_k g_k(y))^2 + g'_k(y)^2$$

Integrating with respect to y over $(\delta, +\infty)$ where $\delta > 0$:

$$(2.5) \quad \int_{\delta}^{\infty} \int_{\Omega} y^{1-2s} |\nabla v(x, y)|^2 dx dy = \sum_{k=1}^{\infty} b_k^2 (-y^{1-2s} g'_k(y) g_k(y)) \Big|_{y=\delta}.$$

From the ODE (2.2) we deduce that $g_k \geq 0$, $g'_k \leq 0$ and $g'_k(y) y^{1-2s}$ is non-decreasing. Thus, if $\delta_i \downarrow 0$, $i \rightarrow \infty$ is a decreasing sequence, $-\delta_i^{1-2s} g'_k(\delta_i) g_k(\delta_i)$ is increasing. By monotone convergence and thanks to (2.4) we deduce

$$\int_0^{\infty} \int_{\mathbb{R}^n} y^{1-2s} |\nabla v(x, y)|^2 dx dy = c_{n,s} \sum_{k=1}^{+\infty} b_k^2 \mu_k^s.$$

This proves that $H \subseteq tr_{\Omega}(H_{0,L}^1(y^{1-2s}))$. \square

Let us remark that if $u \in H$, then the minimization problem

$$\min \left\{ \int_{\mathcal{C}} y^{1-2s} |\nabla v|^2 dx dy : v \in H_{0,L}^1(y^{1-2s}), tr_{\Omega}(v) = u \right\}$$

has a solution $v \in H_{0,L}^1(y^{1-2s})$, by the weak lower semi-continuity of the norm $\| \cdot \|_{H_{0,L}^1(y^{1-2s})}$ and continuity of tr_{Ω} . Moreover the minimizer v is unique, which follows e.g. from the strict convexity of the functional. By standard elliptic theory $v(x, y)$ is smooth for $y > 0$ and satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla v) = 0 & \text{in } \mathcal{C} \\ v = 0 & \text{on } \partial_L \mathcal{C} \\ v = u & \text{on } \Omega \times \{0\} \end{cases}$$

where the boundary condition on $\Omega \times \{0\}$ is in the sense of trace. For each $y > 0$ we may write $v(x, y) = \sum_{k=1}^{\infty} \varphi_k(x) g_k(y)$ where $g_k(y) = \int_{\Omega} v(x, y) dx$. Since $v(\cdot, y) \rightarrow u$ in $L^2(\Omega)$ as $y \rightarrow 0$, $g_k(0)$ are the Fourier coefficients of u , that is $u = \sum_{k=1}^{\infty} g_k(0) \varphi_k$. Then we deduce that $g_k(y)$ is smooth for $y > 0$ and satisfies the ODE (2.2). One can check that $g_k(y) \rightarrow 0$ as $y \rightarrow +\infty$ and therefore $g_k(y) = c_k y^s K_s(\sqrt{\mu_k} y)$ for all $y > 0$ and some $c_k \in \mathbb{R}$. Then, similarly as in (2.5), we obtain for $\delta > 0$

$$(2.6) \quad \int_{\delta}^{\infty} \int_{\mathbb{R}^n} y^{1-2s} |\nabla v(x, y)|^2 dx dy = \sum_{k=1}^{\infty} (-y^{1-2s} g'_k(y) g_k(y)) \Big|_{y=\delta}.$$

Arguing as before, for each k

$$\lim_{y \downarrow 0} (-y^{1-2s} g'_k(y) g_k(y)) = c \mu_k^s g_k(0)^2.$$

We deduce from (2.6) that

$$\|u\|_H^2 = \sum_{k=1}^{\infty} \mu_k^s g_k(0)^2 = c \|v\|_{H_{0,L}^1(y^{1-2s})}^2.$$

In what follows we will call v the *canonical extension* of u .

2.2. Solvability for data in $H^{-s}(\Omega)$. This section is devoted to prove the following lemma:

Lemma 2.2. *Let $h \in H'$. Then, there is a unique $u \in H$ which solves*

$$(2.7) \quad \begin{cases} (-\Delta)^s u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover u is the trace of $v \in H_{0,L}^1(y^{1-2s})$, where v is the unique solution to

$$(2.8) \quad \begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathcal{C} \\ v = 0 & \text{on } \partial_L \mathcal{C} \\ -y^{1-2s}v_y = c_{n,s}h & \text{on } \Omega \times \{0\} \end{cases}$$

where $c_{n,s} > 0$ is a constant depending on n and s only.

Remark. Equation (2.8) is understood in the sense that $v \in H_{0,L}^1(y^{1-2s})$ and

$$(2.9) \quad c_{n,s} \langle h, \operatorname{tr}_\Omega(\zeta) \rangle_{H',H} = \int_{\mathcal{C}} y^{1-2s} \nabla v \cdot \nabla \zeta \, dx dy \quad \text{for all } \zeta \in H_{0,L}^1(y^{1-2s}).$$

The constant $c_{n,s}$ is the same constant appearing in (2.4).

Proof. The case $s = 1/2$ was treated in [8].

The space H' can be identified with the space of distributions $h = \sum_{k=1}^{\infty} h_k \varphi_k$ such that $\sum_{k=1}^{\infty} h_k^2 \mu_k^{-s} < \infty$. Then, it is straight forward to verify that for any $h \in H'$ there is a unique $u \in H$ such that $(-\Delta)^s u = h$. Fix now $h = \varphi_k$ for some $k \geq 1$ and let $u = \mu_k^{-s} \varphi_k$, so that $(-\Delta)^s u = h$. By the Lax-Milgram theorem, there is a unique $v \in H_{0,L}^1(y^{1-2s})$ such that (2.9) holds. Letting g_k denote the unique solution of (2.2)–(2.3), by a direct computation, we find that

$$v(x, y) = \mu_k^{-s} \varphi_k(x) g_k(y)$$

solves (2.9), with $h = \varphi_k$ and its trace is given by $\mu_k^{-s} \varphi_k = u$. This proves the lemma in the case $h = \varphi_k$. By linearity and density, the same holds true for any $h \in H'$. \square

2.3. Maximum principles.

Lemma 2.3. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ any bounded open set. Take $h \in H'$ and let $u \in H$ be the corresponding solution of (2.7). Let also $v \in H_L^1(y^{1-2s})$ denote the canonical extension of u .*

If $h \geq 0$ a.e. in Ω , then $u \geq 0$ a.e. in Ω and $v \geq 0$ in \mathcal{C} .

Proof. Simply use v^- as a test function in (2.9). \square

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ denote any domain and take $R > 0$. Let v denote any locally integrable function on $\Omega \times (0, R)$ such that*

$$\int_{\Omega \times (0, R)} y^{1-2s} |\nabla v|^2 \, dx dy < +\infty.$$

Assume in addition that

$$-\nabla \cdot (y^{1-2s} \nabla v) = 0 \quad \text{in } \Omega \times (0, R),$$

$v \geq 0$ in $\Omega \times (0, R)$, and $-y^{1-2s}v_y|_{y=0} \geq 0$ in Ω in the sense that

$$\int_{\Omega \times (0, R)} y^{1-2s} \nabla v \cdot \nabla \zeta \, dx dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, R))$ such that $\zeta \geq 0$ a.e. in $\Omega \times (0, R)$ and $\zeta = 0$ on $\partial\Omega \times (0, R) \cup \Omega \times \{0\}$.

Then, either $v \equiv 0$, or for any compact subset K of $\Omega \times [0, R)$,

$$\text{ess inf } v|_K > 0.$$

Proof. Let \tilde{v} denote the even extension of v with respect to the y variable, defined in $\Omega \times (-R, R)$ by

$$\tilde{v}(x, y) = \begin{cases} v(x, y) & \text{if } y > 0, \\ v(x, -y) & \text{if } y < 0. \end{cases}$$

Then,

$$\int_{\Omega \times (-R, R)} y^{1-2s} \nabla \tilde{v} \nabla \zeta \, dx dy \geq 0,$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (-R, R))$, such that $\zeta \geq 0$ a.e. in $\Omega \times (-R, R)$ and $\zeta = 0$ on $\partial(\Omega \times (-R, R))$. By the results of Fabes, Kenig, and Serapioni (see Theorem 2.3.1 and the second line of equation (2.3.7) in [14]), either $\tilde{v} \equiv 0$, or $\text{ess inf } \tilde{v}|_K > 0$ for any compact set K of $\Omega \times (-R, R)$. \square

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ denote an open set satisfying an interior sphere condition at some point $x_0 \in \partial\Omega$. Let $R > 0$ and let v denote any measurable function on $\Omega \times (0, R)$, $v \geq 0$, $v \not\equiv 0$, such that*

$$\int_{\Omega \times (0, R)} y^{1-2s} |\nabla v|^2 \, dx dy < +\infty.$$

Assume in addition that

$$-\nabla \cdot (y^{1-2s} \nabla v) = 0 \quad \text{in } \Omega \times (0, R),$$

and $-y^{1-2s} v_y|_{y=0} \geq 0$ in Ω in the sense that

$$\int_{\Omega \times (0, R)} y^{1-2s} \nabla v \cdot \nabla \zeta \, dx dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, R))$ such that $\zeta \geq 0$ a.e. in $\Omega \times (0, R)$ and $\zeta = 0$ on $\partial\Omega \times (0, R) \cup \Omega \times \{0\}$.

Then, there exists $\epsilon > 0$ and a constant $c = c(R) > 0$ such that

$$v(x, y) \geq c|x - x_0| \quad \text{for } x \in B_\epsilon(x_0) \cap \Omega \text{ and } y \in [0, R - 2].$$

Proof. Take an interior sphere B which is tangent to $\partial\Omega$ at x_0 . Translating and dilating Ω if necessary, we may always assume that B is the unit ball centered at the origin. Take $\alpha > n - 2$ to be fixed later and consider $z = z(x, y)$ the function defined by

$$z(x, y) = (1 + y^{2s})(e^{-y^2} - e^{-(R-1)^2})(|x|^{-\alpha} - 1) \quad \text{for } x \neq 0 \text{ and } y \in [0, R - 1].$$

We compute

$$\Delta_x z = (1 + y^{2s})(e^{-y^2} - e^{-(R-1)^2})\alpha(\alpha - (N - 2))|x|^{-\alpha-2},$$

$$\lim_{y \rightarrow 0^+} (-y^{1-2s} z_y) = -2s(1 - e^{-(R-1)^2})(|x|^{-\alpha} - 1) \quad \text{for } x \neq 0,$$

$$z_{yy} + \frac{1-2s}{y} z_y = -4e^{-y^2} [(1-s) + (1+s)y^{2s} - y^2 - y^{2s+2}] (|x|^{-\alpha} - 1).$$

If $y^2 \geq (1+s)$, then $z_{yy} + \frac{1-2s}{y}z_y \geq 0$ and $\nabla \cdot (y^{1-2s}\nabla z) \geq 0$. If $y^2 < (1+s)$, then $z_{yy} + \frac{1-2s}{y}z_y \leq C(|x|^{-\alpha} - 1)$. Choosing α large enough, we deduce that

$$\nabla \cdot (y^{1-2s}\nabla z) \geq 0 \quad \text{for all } x \neq 0, y \in [0, R-1].$$

Now, let v be as in the statement of the lemma. By Lemma 2.4, $\text{ess inf } v|_K > 0$, on $K = \partial B_{1/2} \times [0, R-1]$. Choose $\delta > 0$ so small that $v \geq \delta z$ a.e. on K . By the maximum principle, applied in the region $(B_1 \setminus B_{1/2}) \times (0, R-1)$, we deduce that $v \geq \delta z$ in this region. \square

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let v denote a measurable function on $\Omega \times (0, +\infty)$, such that*

$$\int_{\Omega \times (0, R)} y^{1-2s} |\nabla v|^2 dx dy < +\infty \quad \text{for all } R > 0.$$

Assume that $v \geq 0$ on $\partial\Omega \times (0, +\infty)$, that

$$-\nabla \cdot (y^{1-2s}\nabla v) \geq 0 \quad \text{in } \Omega \times (0, R),$$

and $-y^{1-2s}v_y|_{y=0} \geq 0$ in Ω in the sense that

$$\int_{\Omega \times (0, R)} y^{1-2s} \nabla v \cdot \nabla \zeta dx dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, +\infty))$ with compact support in $\overline{\Omega} \times [0, +\infty)$ such that $\zeta \geq 0$ and $\zeta = 0$ on $\partial\Omega \times (0, R) \cup \Omega \times \{0\}$. If there exist $C > 0$ and $m > 0$ such that

$$(2.10) \quad |v(x, y)| \leq C(1 + |y|^m) \quad \text{for all } (x, y) \in \Omega \times (0, +\infty),$$

then $v \geq 0$ in $\Omega \times (0, +\infty)$.

Proof. Take $R > 0$ such that $\Omega \subseteq B_R(0)$. Let φ_R denote the first eigenfunction of $-\Delta$ in $B_R(0)$ with zero Dirichlet boundary condition and let $\mu_R > 0$ be its corresponding eigenvalue. Let $\lambda > 0$ to be chosen and set

$$z(x, y) = \varphi_R(x)(e^{\lambda y} - \lambda y).$$

We compute

$$\nabla \cdot (y^{1-2s}\nabla z) = y^{1-2s} \left[-\mu_R + \lambda^2 + \lambda^2(1-2s)e^{-\lambda y} \frac{e^{\lambda y} - 1}{\lambda y} \right] \varphi_R(x)e^{\lambda y}.$$

By choosing $\lambda > 0$ small we have $\nabla \cdot (y^{1-2s}\nabla z) < 0$ in $B_R(0) \times (0, +\infty)$. Let $\epsilon > 0$. By (2.10) there exists $L > 0$ such that $v + \epsilon z \geq 0$ for $x \in \Omega$ and $y \geq L$. Using the maximum principle in the form of Lemma 2.3 we deduce that $v + \epsilon z \geq 0$ in $\Omega \times (0, L)$. Letting $L \rightarrow \infty$ we conclude that $v + \epsilon z \geq 0$ in $\Omega \times (0, +\infty)$. Finally, by letting $\epsilon \rightarrow 0$ we obtain the stated result. \square

2.4. Interior regularity. In this section, we study the extension problem (2.8), when h is bounded or belongs to a Hölder space. The proof of the next lemma can be found in [9], Lemma 4.4.

Lemma 2.7. *Let $h \in H'$ and $v \in H_{0,L}^1(y^{1-2s})$ denote the solution of (2.8). Then, for any $\omega \subset\subset \Omega$, $R > 0$, we have*

(i) *If $h \in L^\infty(\Omega)$, then $v \in C^\beta(\omega \times [0, R])$, for any $\beta \in (0, \min(1, 2s))$,*

(ii) If $h \in C^\beta(\Omega)$ then

$$(1) \quad v \in C^{\beta+2s}(\omega \times [0, R]) \quad \text{if } \beta + 2s < 1,$$

$$(2) \quad \frac{\partial v}{\partial x_i} \in C^{\beta+2s-1}(\omega \times [0, R]) \quad \text{if } 1 < \beta + 2s < 2, \quad i = 1, \dots, n,$$

$$(3) \quad \frac{\partial^2 v}{\partial x_i \partial x_j} \in C^{\beta+2s-2}(\omega \times [0, R]) \quad \text{if } 2 < \beta + 2s, \quad i, j = 1, \dots, n.$$

2.5. Boundary regularity.

Lemma 2.8. *Let $u \in H$ be the solution of*

$$(2.11) \quad \begin{cases} (-\Delta)^s u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $h \in L^\infty(\Omega)$. Then $u \in C^\alpha(\bar{\Omega})$ for all $\alpha \in (0, \min(2s, 1))$.

We begin with the following estimate.

Lemma 2.9. *Let $u \in H$ be the solution of (2.11), where $h \in L^\infty(\Omega)$. Then there is constant C such that*

$$\text{if } 0 < s < 1/2, \quad |u(x)| \leq C \text{dist}(x, \partial\Omega)^{2s} \|h\|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega,$$

and

$$\text{if } 1/2 \leq s < 1, \quad |u(x)| \leq C \text{dist}(x, \partial\Omega) \|h\|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega.$$

Proof. We use a suitable barrier to prove the estimate. To construct it, we write $x = (x_1, \dots, x_n)$ and define

$$\bar{h}(x) = \begin{cases} 1 & \text{if } x \in B_2, \quad x_1 < 0 \\ -1 & \text{if } x \in B_2, \quad x_1 > 0 \\ 0 & \text{if } x \notin B_2. \end{cases}$$

We construct a solution \bar{v} of the problem

$$\begin{cases} \text{div}(y^{1-2s} \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty \\ -y^{1-2s} v_y = \bar{h}(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

as

$$(2.12) \quad \bar{v}(x, y) = C_{n,s} \int_y^\infty t \int_{\mathbb{R}^n} \frac{\bar{h}(\tilde{x})}{(t^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{s}}} d\tilde{x} dt.$$

This implies

$$\bar{v}(x, 0) = C'_{n,s} \int_{\mathbb{R}^n} \frac{\bar{h}(\tilde{x})}{|x - \tilde{x}|^{n-2s}} d\tilde{x} \quad x \in \mathbb{R}^n,$$

where $C'_{n,s} = \frac{C_{n,s}}{n-2s}$. By our choice of \bar{h} we can write for $x \in \mathbb{R}^n$

$$\bar{v}(x, 0) = -C'_{n,s} (I(x) - I(-x))$$

where

$$I(x) = \int_{B_2^+} \frac{1}{|x - \tilde{x}|^{n-2s}} d\tilde{x}$$

and $B_2^+ = \{(x_1, \dots, x_n) \in B_2(0) : x_1 > 0\}$. From this formula we see that if $0 < s < 1/2$ then

$$|I(x) - I(0)| \leq C|x|^{2s} \quad \text{for all } x \in \mathbb{R}^n,$$

and if $1/2 \leq s < 1$ then

$$|I(x) - I(0)| \leq C|x| \quad \text{for all } x \in \mathbb{R}^n.$$

These estimates imply that if $0 < s < 1/2$

$$(2.13) \quad |\bar{v}(x)| \leq C|x|^{2s} \quad \text{for all } x \in \mathbb{R}^n,$$

and if $1/2 \leq s < 1$

$$(2.14) \quad |\bar{v}(x)| \leq C|x| \quad \text{for all } x \in \mathbb{R}^n.$$

Now let $u \in H$ be the solution to (2.11) with $h \in L^\infty(\Omega)$ and let v denote its canonical extension. Take a point $x_0 \in \partial\Omega$. By the smoothness of $\partial\Omega$ we can find $x_1 \in \mathbb{R}^n \setminus \Omega$ and $R > 0$ such that $B_R(x_1) \subseteq \mathbb{R}^n \setminus \Omega$ and $x_0 \in \partial B_R(x_1)$. We can choose R bounded and bounded below. By suitable translation and rescaling, we can assume that $x_1 = 0$, $R = 1$ and $|x_0| = 1$. After a further rotation we can also assume $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

We will then define a comparison function w as the Kelvin transform of a translate of \bar{v} as defined by (2.12). Let $\tilde{v}(x, y) = \bar{v}(x - x_0, y)$. We write points in $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ as $X = (x, y)$ and $|X|^2 = |x|^2 + y^2$. We also write \mathbb{R}_+^{n+1} for the set of points $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}$ with $y > 0$. Let

$$w(X) = |X|^{2s-n} \tilde{v}\left(\frac{X}{|X|^2}\right) \quad X \in \mathbb{R}_+^{n+1}, X \neq 0.$$

A direct calculation shows that

$$\operatorname{div}(y^{1-2s} \nabla w) = 0 \quad \text{in } \mathbb{R}_+^{n+1}$$

and

$$\lim_{y \rightarrow 0^+} (-y^{1-2s} w_y(x, y)) = |x|^{-2s-n} \bar{h}\left(\frac{x}{|x|^2}\right) \quad \text{for all } x \in \mathbb{R}^n, x \neq 0.$$

In $\mathbb{R}^n \setminus B_1(0)$ by construction we have $\bar{h}(x/|x|^2) = 1$. Since Ω is bounded and contained in $\mathbb{R}^n \setminus B_1(0)$, we see that there is some constant $c > 0$ (bounded uniformly from below with respect to the parameters x_0, x_1, R with R bounded from below) such that

$$\lim_{y \rightarrow 0^+} (-y^{1-2s} w_y(x, y)) \geq c \quad \text{for all } x \in \Omega.$$

Since $\tilde{v} > 0$ in $B_1(0) \times (0, +\infty)$ we have $w > 0$ in $\Omega \times (0, +\infty)$. Then, there is a constant $c > 0$ (uniformly bounded from below as x_0, x_1 and R vary) such that $w(x, 1) \geq c$ for all $x \in \Omega$. Since $w \geq 0$ on $\partial\Omega \times (0, +\infty)$ and v vanishes there, by the maximum principle we have

$$v \leq C \|h\|_{L^\infty(\Omega)} w \quad \text{in } \Omega \times (0, 1)$$

for some $C > 0$. From this, (2.13) and (2.14) we deduce the stated estimates. \square

Proof of Lemma 2.8. We use a standard scaling argument combined with interior regularity estimates from Lemma 2.7 and Lemma 2.9. Let v denote the canonical extension of u and let us concentrate on the case $0 < s < 1/2$.

Take $x_0, y_0 \in \Omega$. If x_0, y_0 and satisfy $|x_0 - y_0| \geq \text{dist}(x_0, \partial\Omega)/2$ and $|x_0 - y_0| \geq \text{dist}(y_0, \partial\Omega)/2$ from Lemma 2.9

$$\begin{aligned} |v(x_0, 0) - v(y_0, 0)| &\leq |v(x_0, 0)| + |v(y_0, 0)| \leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^{2s} \\ &\leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^\beta. \end{aligned}$$

Now suppose that $|x_0 - y_0| \leq \text{dist}(x_0, \partial\Omega)/2$ and let $r = \text{dist}(x_0, \partial\Omega)/2$. Consider the function $\tilde{v}(x, y) = v(x_0 + rx, ry)$ defined for $x \in B(0, 1)$ and $y > 0$. Thus

$$\text{div}(y^{1-2s}\nabla\tilde{v}) = 0 \quad \text{in } B_1(0) \times (0, +\infty)$$

and

$$\lim_{y \rightarrow 0^+} (-y^{1-2s}\tilde{v}(x, y)) = \tilde{h}(x) \quad x \in B_1(0),$$

where $\tilde{h}(x) = r^{2s}h(rx)$. By Lemma 2.9 we find

$$\sup_{B_1(0)} |\tilde{v}| \leq Cr^{2s}\|h\|_{L^\infty(\Omega)}.$$

Let $0 < \beta < 2s$. Using the interior estimate (Lemma 2.7)

$$\|\tilde{v}\|_{C^\beta(\overline{B}_{1/2})} \leq C(\sup_{B_1} |\tilde{v}| + \sup_{B_1} |\tilde{h}|) \leq Cr^{2s}\|h\|_{L^\infty(\Omega)}$$

we deduce

$$|v(x_0, 0) - v(y_0, 0)| \leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^\beta r^{2s-\beta} \leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^\beta.$$

The proof in the case $1/2 \leq s < 1$ follows analogously. \square

3. PROOF OF PROPOSITION 1.3

We begin by adapting Lemma 1 in [3]:

Lemma 3.1. *Let $n \geq 1$ and let $\Omega \subset \mathbb{R}^n$ denote a smooth bounded domain. Take $f \in L^1(\Omega, \varphi_1 dx)$. Then, there exists a unique $u \in L^1(\Omega, \varphi_1 dx)$ such that*

$$(3.1) \quad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$(3.2) \quad \int_{\Omega} u\psi \, dx = \int_{\Omega} f(-\Delta)^{-s}\psi \, dx, \quad \text{for all } \psi \in C_c^\infty(\Omega).$$

In addition, letting $\mu_1 > 0$ denote the principal eigenvalue of the Laplace operator with homogeneous Dirichlet boundary condition on $\partial\Omega$, we have

$$(3.3) \quad \int_{\Omega} |u|\varphi_1 \, dx \leq \frac{1}{\mu_1} \int_{\Omega} |f|\varphi_1 \, dx.$$

Moreover, if $f \geq 0$ a.e., then $u \geq 0$ a.e. in Ω .

Proof. Take $\psi \in C_c^\infty(\Omega)$. Then, there exists a constant $C > 0$ such that $|\psi| \leq C\varphi_1$. By the maximum principle (Lemma 2.3), it follows that $\varphi = (-\Delta)^{-s}\psi$ satisfies $|\varphi| \leq \frac{C}{\mu_1}\varphi_1$. In particular, (3.2) makes sense for any $\psi \in C_c^\infty(\Omega)$.

Let $f \in L^\infty(\Omega) \subset H'$. Then, equation (3.1) has a unique solution $u \in H$, i.e. for any $\zeta \in H$,

$$\sum_{k=1}^{+\infty} \mu_k^s u_k \zeta_k = \sum_{k=1}^{+\infty} f_k \zeta_k,$$

where $u_k = \int_{\Omega} u \varphi_k \, dx$, and ζ_k, f_k are similarly defined. Take now $\zeta = (-\Delta)^{-s} \psi$, $\psi \in C_c^\infty(\Omega)$. Then, $\zeta_k = \mu_k^{-s} \psi_k$ and

$$\sum_{k=1}^{+\infty} u_k \psi_k = \sum_{k=1}^{+\infty} f_k \mu_k^{-s} \psi_k,$$

which is equivalent to (3.2). We prove next that (3.3) holds. To see this, write $f = f^+ - f^-$, where f^+ is the positive part of f and f^- its negative part. Without loss of generality, we may always assume that $f \geq 0$ a.e. Then, by the maximum principle (Lemma 2.3), $u \geq 0$ a.e. and using (3.2) with $\psi = \varphi_1$, we deduce (3.3). The rest of the proof is the same as that of Lemma 1 in [3], so we skip it. \square

Proof of Proposition 1.3. The method of sub and supersolutions can be applied in the context of solutions of (1.1) belonging to $H \cap L^\infty(B)$. Since $\zeta = 0$ is always a subsolution, we begin by showing that there exists a positive supersolution of (1.1) for small $\lambda > 0$. Take $\zeta_0 \in H$ to be the solution of $(-\Delta)^s \zeta_0 = 1$. By Lemma 2.8, $\zeta_0 \in C(\overline{\Omega})$ and

$$(-\Delta)^s \zeta_0 = 1 \geq \lambda f(\zeta_0), \quad \text{for } \lambda \leq 1/\|f(\zeta_0)\|_{L^\infty(\Omega)}.$$

Hence,

$$\lambda^* = \sup\{\lambda > 0 : (1.1) \text{ has a solution in } H \cap L^\infty(\Omega)\}$$

is positive and well-defined. Multiplying (1.1) by φ_1 and using that f is superlinear, we easily deduce that $\lambda^* < +\infty$. It is also clear by the method of sub and supersolutions that (1.1) has a minimal (hence stable), positive solution $u_\lambda \in H \cap L^\infty(\Omega)$, for all $\lambda \in (0, \lambda^*)$. By minimality, u_λ increases with λ . We claim that $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is a weak solution of (1.1) for $\lambda = \lambda^*$. Take $\lambda < \lambda^*$, $u = u_\lambda$ and multiply (1.1) by φ_1 . Then,

$$(3.4) \quad \mu_1 \int_{\Omega} u \varphi_1 \, dx = \lambda \int_{\Omega} f(u) \varphi_1 \, dx.$$

Since f is superlinear, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that, for all $t \geq 0$ $f(t) \geq \frac{1}{\epsilon} t - C_\epsilon$. Hence,

$$\lambda^* C_\epsilon \geq \left(\frac{\lambda}{\epsilon} - \mu_1 \right) \int_{\Omega} u \varphi_1 \, dx.$$

Choosing $\epsilon = \frac{\lambda}{2\mu_1}$, we obtain that

$$\int_{\Omega} u_\lambda \varphi_1 \, dx \leq C,$$

for some constant C independent of λ . By (3.4), we also have

$$(3.5) \quad \int_{\Omega} f(u_\lambda) \varphi_1 \, dx \leq C,$$

and, by monotone convergence, we may pass to the limit as $\lambda \rightarrow \lambda^*$ in (1.4). \square

Remark. Observe that for $s \geq 1/2$, we have the stronger estimate

$$(3.6) \quad \|u_\lambda\|_{L^1(\Omega)} \leq C,$$

as follows from multiplying (1.1) by ζ_0 and using Lemma 2.9, giving the estimate

$$(3.7) \quad \zeta_0 \leq C \varphi_1.$$

Note also that (3.7) fails for $s < 1/2$. Due to radial monotonicity (see Lemma 4.1), estimate (3.6) remains however true if $\Omega = B_1$ and $s \in (0, 1)$ is arbitrary.

4. RADIAL SYMMETRY

Lemma 4.1. *Let $u \in H \cap L^\infty(B)$ denote a solution of (1.1). Then, u is radially decreasing, i.e. $u(x) = u(\rho)$ whenever $|x| = \rho$, u is smooth in B , and*

$$(4.1) \quad \frac{\partial u}{\partial \rho} < 0 \quad \text{in } B \setminus \{0\}.$$

In addition, the canonical extension v of u is smooth in \mathcal{C} , $v(x, y) = v(\rho, y)$, and

$$(4.2) \quad \frac{\partial v}{\partial \rho} < 0 \quad \text{in } \mathcal{C} \setminus \{\rho = 0\}.$$

Proof. The smoothness of u and v follows from Lemmata 2.7 and 2.8. To prove radial symmetry, (4.1), and (4.2), we apply the moving plane method ([16]). Thus, it suffices to show that

$$\frac{\partial v}{\partial x_1} < 0 \quad \text{in } \{(x, y) \in B_1 \times [0, +\infty) : x_1 > 0\}.$$

Now we show the last statement. Given $\mu \in (0, 1]$, let $T_\mu = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^+ : x_1 = \mu\}$ and $\Sigma_\mu = \{(x, y) \in B_1 \times [0, +\infty) : x_1 > \mu\}$. Let also $v_\mu(x, y) = v(2\mu - x_1, x', y)$ for $(x, y) \in \Sigma_\mu$ and $w_\mu = v_\mu - v$. We claim that $w_\mu \geq 0$ in Σ_μ , for μ close to 1. To prove this, observe that $w = w_\mu$ solves

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \Sigma_\mu \\ w \geq 0 & \text{on } \partial_L \Sigma_\mu \\ -y^{1-2s}w_y - a(x)w = 0 & \text{on } \{x \in B_1 : x_1 > \mu\} \times \{0\}, \end{cases}$$

where

$$(4.3) \quad a(x) = \begin{cases} \frac{f(u_\mu) - f(u)}{u_\mu - u} & \text{whenever } u_\mu \neq u, \\ 0 & \text{otherwise.} \end{cases}$$

Now multiply the above equation by w^- and integrate over Σ_μ . Then,

$$\int_{\Sigma_\mu} y^{1-2s} |\nabla w^-|^2 dx dy = \int_{\{x \in B_1 : x_1 > \mu\}} a(x) (w^-)^2 dx.$$

We extend w^- by 0 outside Σ_μ , so that $w^- \in H^1(y^{1-2s}; \mathbb{R}^n)$. By the trace theorem (Proposition 2.1), there exists a constant $C_{tr} > 0$ such that

$$\|w^-\|_{H^s(\mathbb{R}^n)}^2 \leq C_{tr} \int_{\mathbb{R}^n} y^{1-2s} |\nabla w^-|^2 dx dy,$$

and by the Sobolev imbedding of $H^s(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, with

$$(4.4) \quad \frac{1}{p} = \frac{1}{2} - \frac{s}{n},$$

we have

$$\|w^-\|_{L^p(\mathbb{R}^n)}^2 \leq C_S \|w^-\|_{H^s(\mathbb{R}^n)}^2.$$

Hence, by Hoelder's inequality

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |w^-|^p dx \right)^{2/p} &\leq C_{tr} C_S \int_{\{x \in B_1 : x_1 > \mu\}} |a(x)| (w^-)^2 dx \\ &\leq C_{tr} C_S \left(\int_{\{x \in B_1 : x_1 > \mu\}} (w^-)^p dx \right)^{2/p} \left(\int_{\{x \in B_1 : x_1 > \mu\}} |a|^{\frac{p}{p-2}} dx \right)^{1-2/p} \end{aligned}$$

Since a is uniformly bounded, $\int_{\{x \in B_1 : x_1 > \mu\}} |a|^{\frac{p}{p-2}} dx \rightarrow 0$, as $\mu \rightarrow 1^-$. Therefore, for μ sufficiently close to 1, we conclude that $w^- \equiv 0$, and the claim.

Consider now

$$\mu_0 = \inf \{ \mu \in (0, 1) : w_\mu \geq 0 \text{ in } \Sigma_\mu \}.$$

The above argument shows that μ_0 is well-defined and $\mu_0 < 1$. We want to prove that $\mu_0 = 0$. Assume by contradiction that $\mu_0 > 0$. By continuity, $w_{\mu_0} \geq 0$ in Σ_{μ_0} , and by the strong maximum principle (Lemma 2.4), $w_{\mu_0} > 0$ in Σ_{μ_0} . Fix now $\epsilon > 0$ small, $\mu = \mu_0 - \epsilon$ and choose a compact set $K \subset \{x \in B_1 : x_1 > \mu_0\}$ such that

$$C_{tr} C_S \left(\int_{\{x \in B_1 : x_1 > \mu\} \setminus K} |a|^{\frac{p}{p-2}} dx \right)^{1-2/p} < \frac{1}{2}.$$

Taking $\epsilon > 0$ smaller if necessary, we can assume that $w_\mu > 0$ in K . Arguing as before, we can prove that $w_\mu^- \equiv 0$ in $\Sigma_\mu \setminus K$, and thus $w_\mu \geq 0$ everywhere in Σ_μ , contradicting the definition of μ_0 .

We have just proved that $w_\mu \geq 0$ in Σ_μ for all $\mu \in (0, 1)$, and by the strong maximum principle (Lemma 2.4) we find that $w_\mu > 0$ in Σ_μ . Finally, by the boundary point lemma (Lemma 2.5), we conclude

$$2 \frac{\partial v_\mu}{\partial x_1}(\mu, x', y) = - \frac{\partial w_\mu}{\partial x_1}(\mu, x', y) < 0 \quad \text{for all } (\mu, x', y) \in B_1 \times [0, +\infty),$$

as desired. \square

5. WEIGHTED INTEGRABILITY

We will use the following notation. Given a point $(x, y) \in \mathcal{C} = B_1 \times (0, +\infty)$, we let $\rho = |x|$ and $v_\rho = \frac{\partial v}{\partial \rho}$ for any C^1 function v defined on \mathcal{C} , which depends only on ρ and y .

In what follows, for $\lambda \in [0, \lambda^*)$, u_λ denotes the minimal solution of (1.1) and v_λ its canonical extension, which satisfies

$$(5.1) \quad \begin{cases} \operatorname{div}(y^{1-2s} \nabla v) = 0 & \text{in } \mathcal{C} \\ v = 0 & \text{on } \partial_L \mathcal{C} \\ -y^{1-2s} v_y = \lambda f(v) & \text{on } B_1 \times \{0\}. \end{cases}$$

By elliptic regularity (Lemmata 2.7 and 2.8), for $\lambda \in [0, \lambda^*)$, $u_\lambda \in C^\infty(B_1) \cap C(\overline{B_1})$, and v_λ is smooth in \mathcal{C} . By Lemma 2.7, we also deduce that $v_\lambda \in C^\alpha(K \times [0, R])$ for every compact $K \subset B_1$ and $R > 0$. Moreover, any of the derivatives of v_λ with respect to the x variables belongs to $C^\alpha(K \times [0, R])$ for every compact set $K \subset B_1$ and $R > 0$.

The main result in this section is the following.

Proposition 5.1. *Assume $n \geq 2$. Let $\lambda \in (0, \lambda^*)$, $u = u_\lambda$ be the minimal solution of (1.1) and v its canonical extension. Let α satisfy*

$$(5.2) \quad 1 \leq \alpha < 1 + \sqrt{n-1}.$$

Then

$$(5.3) \quad \int_{[\rho \leq 1/2]} y^{1-2s} v_\rho^2 \rho^{-2\alpha} dx dy \leq C$$

where C is a constant independent of λ , and $[\rho \leq 1/2]$ denotes the set $\{(x, y) \in \mathcal{C} : |x| \leq 1/2\}$.

We collect in the next lemma some basic estimates expressing that v_λ and its derivatives have exponential decay for $y \geq 1$, which is uniform up to $\lambda < \lambda^*$, and that for fixed $\lambda < \lambda^*$, $v_\rho(\rho, y) = O(\rho)$ as $\rho \rightarrow 0$, uniformly as $y \rightarrow 0$.

Lemma 5.2. *a) There are $\gamma > 0$, $C > 0$ such that*

$$(5.4) \quad v_\lambda(x, y) \leq C e^{-\gamma y} \varphi_1(x) \quad \text{for all } y \geq 1, x \in B_1, \lambda \in [0, \lambda^*].$$

Moreover, for any $k \geq 0$ there is $C_k > 0$ such that

$$(5.5) \quad |D^k v_\lambda(x, y)| \leq C_k e^{-\gamma y} \quad \text{for all } y \geq 1, x \in B_1, \lambda \in [0, \lambda^*].$$

The constants γ and C are independent of λ .

b) Given $\lambda \in [0, \lambda^)$ and K a compact subset of B_1 there exists $C > 0$ such that*

$$(5.6) \quad |\partial_\rho v_\lambda(|x|, y)| \leq C|x| \quad \forall x \in K, y \geq 0.$$

Proof. a) Define $w(x, y) = \varphi_1(x) y^{2s} e^{-\gamma y}$. A straight forward computation shows that

$$\nabla(y^{1-2s} \nabla w) = \varphi_1(x) e^{-\gamma y} [(\gamma^2 - \lambda_1)y - \gamma(1 + 2s)]$$

and

$$y^{1-2s} w_y \Big|_{y=0} = \lim_{y \rightarrow 0} y^{1-2s} \varphi_1(x) e^{-\gamma y} (-\gamma y^{2s} + 2s y^{2s-1}) = 2s \varphi_1(x).$$

Multiplying equation (5.1) by w and integrating by parts twice gives

$$\lambda \int_{B_1} f(u_\lambda) w dx + \int_{B_1} y^{1-2s} w_y v_\lambda dx + \int_{\mathcal{C}} \nabla(y^{1-2s} \nabla w) v_\lambda = 0.$$

Recalling that $w(x, 0) = 0$, we find

$$2s \int_{B_1} \varphi_1 u_\lambda dx = \int_{\mathcal{C}} v_\lambda \varphi_1(x) e^{-\gamma y} [(\lambda_1 - \gamma^2)y + \gamma(1 + 2s)] dx dy.$$

Now, we choose $0 < \gamma < \sqrt{\lambda_1}$ and use estimate $\int_{B_1} \varphi_1 u_\lambda dx \leq C$ derived in (3.5), to find

$$(5.7) \quad \int_{\mathcal{C}} v_\lambda \varphi_1(x) e^{-\gamma y} dx dy \leq C$$

for all $0 \leq \lambda < \lambda^*$.

Let z be the solution to

$$\begin{cases} -\Delta z = 1 & \text{in } B_1 \\ z = 0 & \text{on } \partial B_1. \end{cases}$$

For $\tau \geq t > 0$ define $\varphi(x, y) = z(x)(\tau - y)(y - t)$. We compute

$$\nabla(y^{1-2s} \nabla \varphi) = y^{1-2s} \left[-(\tau - y)(y - t) + z(x) \left(-2 + (1 - 2s) \left(-2 + \frac{\tau + t}{y} \right) \right) \right]$$

Assume that $0 < t \leq \tau \leq 3t/2$. We find

$$\nabla(y^{1-2s}\nabla\varphi) \leq -y^{1-2s}(\tau - y)(y - t).$$

Multiplying (5.1) by φ and integrating over $B_1 \times (t, \tau)$ we obtain

$$\begin{aligned} & \tau^{1-2s}(\tau - t) \int_{B_1} v_\lambda(x, \tau) z(x) dx + t^{1-2s}(\tau - t) \int_{B_1} v_\lambda(x, t) z(x) dx \\ &= - \int_{B_1 \times (t, \tau)} y^{1-2s} v_\lambda \nabla(y^{1-2s} \nabla \varphi) dx dy \geq \int_{B_1 \times (t, \tau)} y^{1-2s} v_\lambda (\tau - y)(y - t) dx dy. \end{aligned}$$

Thus, for $t \geq 6$ we deduce

$$\begin{aligned} \int_{B_1 \times (t+1, t+2)} y^{1-2s} v_\lambda dx dy &\leq C t^{1-2s} \int_{B_1} v_\lambda(x, t) z(x) dx \\ &\quad + C(t+3)^{1+2s} \int_{B_1} v_\lambda(x, t+3) z(x) dx. \end{aligned}$$

Integrating this inequality with respect to $t \in [6, 13]$, recalling that $z \leq C\varphi_1$ for some $C > 0$, and using (5.7) we obtain

$$\int_{B_1 \times [8, 11]} v_\lambda dx dy \leq C$$

with a constant independent of λ as $\lambda \rightarrow \lambda^*$.

This inequality and standard elliptic estimates imply

$$(5.8) \quad v_\lambda(x, y) \leq C e^{-\gamma y} \varphi_1(x) \quad \text{for all } y \in [9, 10], x \in B_1, \text{ and } \lambda \in [0, \lambda^*).$$

Now let $\bar{w}(x, y) = C\varphi_1(x)e^{-\gamma y}$. For $0 < \gamma < \sqrt{\lambda_1}$, this is a supersolution of the equation in (5.1) and by comparison in $B_1 \times (1, +\infty)$, using (5.8), we deduce (5.4). Inequality (5.5) is a consequence of (5.4) and elliptic estimates.

b) This part follows from the fact that for $\lambda < \lambda^*$, u_λ is smooth in B_1 and hence v_λ and its derivatives with respect to the x variables are in $C^\alpha(K \times [0, R])$ for any compact $K \subset B_1$ and $R > 0$. \square

The following result is a version of Lemma 1 of [6] in the case of radially symmetric functions.

Lemma 5.3. *Given $\lambda \in (0, \lambda^*)$, let $u = u_\lambda \in H \cap L^\infty(B_1)$ denote the minimal solution of (1.1), and let $v \in H_{0,L}^1(y^{1-2s})$ denote its canonical extension. Then, for every $\eta \in C^1(B_1 \times [0, +\infty))$ with compact support in \mathcal{C} , but not necessarily vanishing on $B_1 \times \{0\}$, we have*

$$(5.9) \quad \int_{\mathcal{C}} y^{1-2s} v_\rho^2 |\nabla \eta|^2 dx dy \geq (n-1) \int_{\mathcal{C}} y^{1-2s} \frac{v_\rho^2}{\rho^2} \eta^2 dx dy.$$

Proof. Inequality (1.5) implies that for all $\xi \in H_{0,L}^1(y^{1-2s})$, there holds

$$(5.10) \quad \int_{\mathcal{C}} y^{1-2s} |\nabla \xi|^2 dx dy \geq \int_{B_1} f'(u) \xi^2 dx,$$

where in the right-hand-side integral we identified ξ and its trace.

Let $\eta \in C^1(B_1 \times [0, +\infty))$ as in the statement of the lemma and take $\xi = \eta v_\rho$. By Lemma 5.2, $\xi \in H_{0,L}^1(y^{1-2s})$ and from (5.10) we obtain

$$\begin{aligned}
 \lambda \int_{B_1} f'(u) v_\rho^2 \eta^2 dx &\leq \int_{\mathcal{C}} y^{1-2s} |\nabla(v_\rho \eta)|^2 dx dy \\
 &\leq \int_{\mathcal{C}} y^{1-2s} \{ |\nabla v_\rho|^2 \eta^2 + v_\rho^2 |\nabla \eta|^2 + v_\rho \nabla v_\rho \cdot \nabla \eta^2 \} dx dy \\
 (5.11) \quad &\leq \int_{\mathcal{C}} y^{1-2s} \{ v_\rho^2 |\nabla \eta|^2 + \nabla(\eta^2 v_\rho) \cdot \nabla v_\rho \} dx dy.
 \end{aligned}$$

Since by Lemma 4.1, u is radially symmetric, by differentiation of (5.1) with respect to ρ , one gets

$$(5.12) \quad \nabla \cdot (y^{1-2s} \nabla v_\rho) = y^{1-2s} \frac{n-1}{\rho^2} v_\rho \quad \text{in } \mathcal{C}.$$

Next, we differentiate the Neumann boundary condition in (5.1) with respect to ρ to obtain

$$(5.13) \quad -y^{1-2s} \partial_y v_\rho = \lambda f'(v) v_\rho \quad \text{for } 0 \leq \rho < 1.$$

Now, we multiply (5.12) by $\eta^2 v_\rho$, and integrate by parts and use (5.13) to find

$$\int_{\mathcal{C}} y^{1-2s} \nabla(\eta^2 v_\rho) \cdot \nabla v_\rho dx dy = \lambda \int_{B_1} f'(u) (v_\rho \eta)^2 dx - (n-1) \int_{\mathcal{C}} y^{1-2s} \frac{(v_\rho \eta)^2}{\rho^2} dx dy.$$

Combining the last equation with (5.11) yields (5.9). \square

Proof of Proposition 5.1. Given $\varepsilon > 0$ let $\zeta_\varepsilon \in C^\infty(\mathbb{R})$ be such that $\zeta_\varepsilon(t) = 0$ for $t \leq \varepsilon$ and $t \geq 3/4$, $\zeta_\varepsilon(t) = 1$ for $t \in [2\varepsilon, 1/2]$, and $\zeta(t) \leq C/\varepsilon$ for $t \in [\varepsilon, 2\varepsilon]$. Given $R > 0$ we let ϕ_R denote a function $C^\infty(\mathbb{R})$ such that $\psi_R(y) = 1$ for all $r \leq R$ and $\psi_R(y) = 0$ for all $y \geq R+1$.

Let α satisfy (5.2) and for $\varepsilon > 0$, $R > 0$ define $\eta(\rho, y) = \rho^{1-\alpha} \zeta_\varepsilon(\rho) \psi_R(y)$. Given $\delta > 0$ we estimate

$$|\nabla \eta|^2 \leq ((1-\alpha)^2 + \delta) \rho^{-2\alpha} \zeta_\varepsilon(\rho)^2 \psi_R(y)^2 + C_\delta \rho^{2-2\alpha} |\nabla(\zeta_\varepsilon \psi_R)|^2$$

for some $C_\delta > 0$. Then by (5.9)

$$\begin{aligned}
 (n-1) \int_{\mathcal{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx dy &\leq ((1-\alpha)^2 + \delta) \int_{\mathcal{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx dy \\
 &\quad + C \int_{\mathcal{C}} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 |\nabla(\zeta_\varepsilon \psi_R)|^2 dx dy.
 \end{aligned}$$

Choosing $\delta > 0$ small enough

$$\int_{\mathcal{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx dy \leq C \int_{\mathcal{C}} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 (|\nabla \zeta_\varepsilon|^2 \psi_R^2 + \zeta_\varepsilon^2 |\nabla \psi_R|^2) dx dy$$

where $C > 0$. Thanks to (5.6) we have

$$\begin{aligned}
 \int_{\mathcal{C}} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 |\nabla \zeta_\varepsilon|^2 \psi_R^2 dx dy &\leq \frac{C}{\varepsilon^2} \int_{[\varepsilon \leq \rho \leq 2\varepsilon, 0 \leq y \leq R+1]} y^{1-2s} \rho^{4-2\alpha} dx dy \\
 &\leq C(R+1)^{2-2s} \varepsilon^{2-2\alpha+n}.
 \end{aligned}$$

Because of (5.2) we have that $2 - 2\alpha + n > 0$. Letting $\varepsilon \rightarrow 0$ we find

$$\int_{[\rho \leq 1/2, y \leq R]} y^{1-2s} v_\rho^2 \rho^{-2\alpha} dx dy \leq C \int_{[1/2 \leq \rho \leq 3/4] \cup [R \leq y \leq R+1]} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 dx dy \leq C$$

where the last inequality follows from (5.5). Finally, letting $R \rightarrow \infty$ we conclude (5.3). \square

For $0 < \beta < n$ we define

$$(5.14) \quad A_{n,s,\beta} = \int_{\mathbb{R}^n \times (0, +\infty)} \frac{y^{3-2s}}{(|x|^2 + y^2)^{\frac{\beta+2}{2}} (y^2 + |x-e|^2)^{\frac{n+2-2s}{2}}} dx dy$$

where $r = |(x, y)| = (\rho^2 + y^2)^{1/2}$, and e is any unit vector in \mathbb{R}^n .

Lemma 5.4. *We have $1 - \beta C_{n,s} A(n, s, \beta) > 0$, where $C_{n,s}$ is the constant in the representation formula (2.12).*

Proof. Let $h \in L^\infty(\mathbb{R}^n)$ be radial and have compact support, and $u(x, y) = u(\rho, y)$ be a solution of

$$(5.15) \quad \begin{cases} \operatorname{div}(y^{1-2s} \nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x, y) \rightarrow 0 & \text{as } |(x, y)| \rightarrow \infty \\ -y^{1-2s} u_y = h(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Now, we claim that, for any $0 < \beta < n$

$$(5.16) \quad 0 = (1 - \beta C_{n,s} A_{n,s,\beta}) \int_{\mathbb{R}^n} h(x) \rho^{-\beta} dx + \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s} r^{-\beta-2} \rho u_\rho dx dy.$$

Assuming the claim for a moment we prove the lemma. Choose a smooth radially decreasing function $h \geq 0$, $h \not\equiv 0$ with compact support. Let u be the solution of (5.15). By (2.12), u can be explicitly given by a convolution kernel. In turn, this shows that u is radial with respect to x and non-increasing in $|x|$. Hence

$$\int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s} r^{-\beta-2} \rho u_\rho dx dy < 0$$

and

$$\int_{\mathbb{R}^n} h(x) \rho^{-\beta} dx > 0.$$

This shows that $1 - \beta C_{n,s} A_{n,s,\beta} > 0$.

Now we give the argument for (5.16). Let $\varepsilon > 0$, $\beta \in (0, n + 2 - 2s)$ and multiply equation (5.15) by $(\rho^2 + y^2 + \varepsilon)^{-\beta/2}$ to get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n \times (0, +\infty)} \operatorname{div}(y^{1-2s} \nabla u) (\rho^2 + y^2 + \varepsilon)^{-\beta/2} dx dy \\ &= - \int_{\mathbb{R}^n} y^{1-2s} u_y (\rho^2 + \varepsilon)^{-\beta/2} dx + \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} (x \cdot \nabla_x u + y u_y) dx dy \\ &= \int_{\mathbb{R}^n} h(x) (\rho^2 + \varepsilon)^{-\beta/2} dx + \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} \rho u_\rho dx dy \\ &\quad + \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{2-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} u_y dx dy \end{aligned}$$

Using the representation formula

$$-y^{1-2s}u_y(x, y) = C_{n,s}y^{2-2s} \int_{\mathbb{R}^n} \frac{h(\tilde{x})}{(y^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{2}}} d\tilde{x}$$

we find

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} h(x)(\rho^2 + \varepsilon)^{-\beta/2} dx + \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} \rho u_\rho dx dy \\ &\quad - \beta C_{n,s} \int_{\mathbb{R}^n \times (0, +\infty)} \int_{\mathbb{R}^n} y^{3-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} \frac{h(\tilde{x})}{(y^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{2}}} d\tilde{x} dx dy. \end{aligned}$$

By Fubini, the last integral becomes

$$\int_{\mathbb{R}^n} h(\tilde{x}) \int_{\mathbb{R}^n \times (0, +\infty)} \frac{y^{3-2s}}{(|x|^2 + y^2 + \varepsilon)^{\beta/2+1} (y^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{2}}} dx dy d\tilde{x},$$

and by the change variables: $y = |\tilde{x}|y'$, $y > 0$, $x = |\tilde{x}|x'$, $x' \in \mathbb{R}^n$, we find

$$\int_{\mathbb{R}^n \times (0, +\infty)} \frac{y^{3-2s}}{(|x|^2 + y^2 + \varepsilon)^{\beta/2+1} (y^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{2}}} dx dy = |\tilde{x}|^{-\beta} A_{n,s,\beta} \left(\frac{\varepsilon}{|\tilde{x}|^2} \right)$$

where

$$A_{n,s,\beta}(t) = \int_{\mathbb{R}^n \times (0, +\infty)} \frac{y^{3-2s}}{(|x|^2 + y^2 + t)^{\frac{\beta+2}{2}} (y^2 + |x - \frac{\tilde{x}}{|\tilde{x}|}|^2)^{\frac{n+2-2s}{2}}} dx dy.$$

Therefore, from the above computations we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} h(x)(\rho^2 + \varepsilon)^{-\beta/2} (1 - \beta C_{n,s} A_{n,s,\beta}(\varepsilon/|\tilde{x}|^2)) dx \\ (5.17) \quad &+ \beta \int_{\mathbb{R}^n \times (0, +\infty)} y^{1-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} \rho u_\rho dx dy \end{aligned}$$

Notice that

$$\lim_{\varepsilon \rightarrow 0} A_{n,s,\beta}(\varepsilon/|\tilde{x}|^2) = A_{n,s,\beta} \quad \text{for all } \tilde{x} \in \mathbb{R}^n$$

and that this limit is finite for $0 < \beta < n + 2 - 2s$. Moreover $A_{n,s,\beta}$ is independent of \tilde{x} . Since $\beta < n$ and h is bounded with compact support the function $h(\rho)\rho^{-\beta}$ is integrable. Hence, by letting $\varepsilon \rightarrow 0$ in (5.17) we obtain (5.16). \square

6. PROOF OF THEOREM 1.4

Lemma 6.1. *Let $h \in L^\infty(B_1)$ and $u \in H$ be the unique solution of*

$$(-\Delta)^s u = h \quad \text{in } B_1.$$

Then

$$(6.1) \quad |u(x)| \leq C_{n,s} \int_{B_1} \frac{|h(\tilde{x})|}{|x - \tilde{x}|^{n-2s}} d\tilde{x} \quad \text{for every } x \in B_1.$$

Proof. Writing $h = h^+ - h^-$ with $h^+, h^- \geq 0$ we see that it is sufficient to prove the result in the case $h \geq 0$, so that also $u \geq 0$.

Let v be the canonical extension of u . Since $v(x, \infty) = 0$, for every x , we can write

$$(6.2) \quad v(x, 0) = - \int_0^\infty v_y(x, y) dy \quad \text{for all } x \in B_1.$$

Let $g(x)$ be equal to $h(x)$ extended by 0 in $\mathbb{R}^n \setminus B_1$, and denote by \tilde{v} the solution of

$$(6.3) \quad \begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{v}) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \tilde{v}(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty \\ -y^{1-2s}\tilde{v}_y = g(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

By the Green's representation formula for (6.3), we have

$$(6.4) \quad -\tilde{v}_y(x, y) = C_{n,s} y \int_{\mathbb{R}^n} \frac{g(\tilde{x})}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} d\tilde{x}.$$

Consider the functions $w = -y^{1-2s}v_y$ and $\tilde{w} = -y^{1-2s}\tilde{v}_y$. Then, w and \tilde{w} satisfy

$$\nabla(y^{2s-1}\nabla w) = 0 \quad \text{in } \mathcal{C}.$$

Since $-\tilde{v}_y \geq 0$ in $\mathbb{R}^n \times [0, +\infty)$ in particular we have

$$\tilde{w} \geq 0 = w \quad \text{on } \partial_L \mathcal{C}.$$

Furthermore

$$w \leq \tilde{w} \quad \text{in } B_1 \times \{0\}$$

and for $z \in \mathcal{C}$, $w(z), \tilde{w}(z) \rightarrow 0$ as $|z| \rightarrow +\infty$. Then, the maximum principle (Lemma 2.6) implies that

$$(6.5) \quad -v_y \leq -\tilde{v}_y \quad \text{in } \mathcal{C}.$$

Combining (6.2), (6.5) together with (6.4) we find

$$\begin{aligned} v(x, 0) &\leq C_{n,s} \int_0^\infty y \int_{\mathbb{R}^n} \frac{g(\tilde{x})}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} d\tilde{x} dy \\ &= C_{n,s} \int_{\mathbb{R}^n} g(\tilde{x}) \left(\int_0^\infty \frac{y}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} dy \right) d\tilde{x} \end{aligned}$$

for all $x \in B_1$, where we have used Fubini's theorem in the last line. Claim (6.1) follows by performing the integration over the y variable in the last expression, and recalling the definition of $g(x)$. \square

Proof of Theorem 1.4. We denote points in $\mathcal{C} = B_1 \times (0, +\infty)$ as $(x, y) \in \mathcal{C}$, where $x \in B_1$, $y \in (0, +\infty)$, and $\rho = |x|$.

Step 1. Take α such that (5.2) holds. We claim that for $\beta > 0$ such that $2(\beta + s - \alpha) < n$ we have

$$(6.6) \quad \int_{B_1} f(u_\lambda) \rho^{-\beta} dx \leq C$$

with C independent of λ as $\lambda \rightarrow \lambda^*$.

To prove the claim, let $\varepsilon > 0$, $R > 0$ and multiply (2.8) by $(\rho^2 + y^2 + \varepsilon)^{-\beta/2}$ and integrate over $[\rho \leq 1/2, 0 \leq y \leq R]$ to get

$$0 = \int_{[\rho \leq 1/2, 0 \leq y \leq R]} \nabla \cdot (y^{1-2s}\nabla v)(\rho^2 + y^2 + \varepsilon)^{-\beta/2} dx dy.$$

Integrating by parts we find

$$(6.7) \quad \lambda \int_{B_{1/2}} f(u_\lambda)(\rho^2 + \varepsilon)^{-\beta/2} dx = -I_1 - I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_{[\rho \leq 1/2]} R^{1-2s} v_y(\rho, R) (\rho^2 + R^2 + \varepsilon)^{-\beta/2} dx \\ I_2 &= \int_{[0 \leq y \leq R]} y^{1-2s} v_\rho(1/2, y) (1/4 + y^2 + \varepsilon)^{-\beta/2} dy \\ I_3 &= -\beta \int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} (v_\rho \rho + v_y y) dx dy. \end{aligned}$$

By (5.4) and (5.5), I_1 and I_2 remain uniformly bounded as $\varepsilon \rightarrow 0$ and $\lambda \rightarrow \lambda^*$. We decompose further

$$I_3 = I_\rho + I_y$$

where

$$\begin{aligned} I_\rho &= -\beta \int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} v_\rho \rho dx dy, \\ I_y &= -\beta \int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} v_y y dx dy. \end{aligned}$$

Now we estimate I_y . Let $g(x)$ be equal to $\lambda f(u_\lambda(x))$ extended by 0 in $\mathbb{R}^n \setminus B_1$, and denote by \tilde{v} the solution of

$$(6.8) \quad \begin{cases} \operatorname{div}(y^{1-2s} \nabla \tilde{v}) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \tilde{v}(z) \rightarrow 0 & \text{as } |z| \rightarrow \infty \\ -y^{1-2s} \tilde{v}_y = g(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

By the Green representation formula for (6.8), we have

$$(6.9) \quad -\tilde{v}_y(x, y) = C_{n,s} y \int_{\mathbb{R}^n} \frac{g(\tilde{x})}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} d\tilde{x}.$$

Consider the functions $w = -y^{1-2s} v_y$ and $\tilde{w} = -y^{1-2s} \tilde{v}_y$. Then, w and \tilde{w} satisfy

$$\nabla \cdot (y^{2s-1} \nabla w) = 0 \quad \text{in } \mathcal{C}.$$

Since $-\tilde{v}_y \geq 0$ in $\mathbb{R}^n \times [0, +\infty)$ we have in particular

$$\tilde{w} \geq 0 = w \quad \text{on } \partial_L \mathcal{C}.$$

Furthermore

$$w \leq \tilde{w} \quad \text{in } B_1 \times \{0\}$$

and for $z \in \mathcal{C}$, $w(z), \tilde{w}(z) \rightarrow 0$ as $|z| \rightarrow +\infty$. Then, the maximum principle (Lemma 2.6) implies that

$$-v_y \leq -\tilde{v}_y \quad \text{in } \mathcal{C}.$$

It follows that

$$I_y \leq -\beta \int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} (\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} \tilde{v}_y y dx dy$$

and by (6.9)

$$\begin{aligned} I_y &\leq \beta C_{n,s} \int_{[\rho \leq 1/2, 0 \leq y \leq R]} \int_{\mathbb{R}^n} \frac{y^{3-2s} g(\tilde{x})}{(\rho^2 + y^2 + \varepsilon)^{\beta/2+1} (|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} d\tilde{x} dx dy \\ &\leq \beta C_{n,s} \int_{\mathbb{R}^n} g(\tilde{x}) \left(\int_{[\rho \leq 1/2, 0 \leq y \leq R]} \frac{y^{3-2s}}{(\rho^2 + y^2 + \varepsilon)^{\beta/2+1} (|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} dx dy \right) d\tilde{x}. \end{aligned}$$

Therefore

$$(6.10) \quad I_y \leq \beta C_{n,s} A_{n,s,\beta} \int_{\mathbb{R}^n} g(\tilde{x}) d\tilde{x} = \beta C_{n,s} A_{n,s,\beta} \lambda \int_{B_1} f(u_\lambda) dx$$

where $A_{n,s,\beta}$ is defined as in (5.14). Combining (6.7) with (6.10) we obtain

$$(6.11) \quad (1 - \beta C_{n,s} A_{n,s,\beta}) \lambda \int_{B_1} f(u_\lambda) dx \leq \lambda \int_{B_1 \setminus B_{1/2}} f(u_\lambda) dx - I_1 - I_2 + I_\rho.$$

Recall that by (5.4) and (5.5),

$$(6.12) \quad |I_1| \leq C, \quad |I_2| \leq C$$

for some C independent $\varepsilon \rightarrow 0$ and $\lambda \rightarrow \lambda^*$.

By the Cauchy-Schwarz inequality

$$|I_\rho| \leq \beta \left(\int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} v_\rho^2 \rho^{-2\alpha} dx dy \right)^{1/2} \left(\int_{[\rho \leq 1/2, 0 \leq y \leq R]} \frac{y^{1-2s} \rho^{2+2\alpha}}{(\rho^2 + y^2 + \varepsilon)^{\beta+2}} dx dy \right)^{1/2}$$

The last integral can be estimated by

$$\int_{[\rho \leq 1/2, 0 \leq y \leq R]} \frac{y^{1-2s} \rho^{2+2\alpha}}{(\rho^2 + y^2 + \varepsilon)^{\beta+2}} dx dy \leq \int_0^\infty \int_{[\rho \leq 1/2]} \frac{y^{1-2s} \rho^{2\alpha}}{(\rho^2 + y^2)^{\beta+1}} dx dy$$

We change variables $y = \rho t$ for $\rho > 0$. Since $\beta > 0$, we have

$$\int_{[\rho \leq 1/2, 0 \leq y \leq R]} \frac{y^{1-2s} \rho^{2+2\alpha}}{(\rho^2 + y^2 + \varepsilon)^{\beta+2}} dx dy \leq \int_{[\rho \leq 1/2]} \rho^{2\alpha-2\beta-2s} dx \int_0^\infty \frac{t^{1-2s}}{(1+t^2)^{\beta+1}} dt$$

and this integral is finite if $2(\beta+s-\alpha) < n$. The integral $\int_{[\rho \leq 1/2, 0 \leq y \leq R]} y^{1-2s} v_\rho^2 \rho^{-2\alpha} dx dy$ remains bounded as $\varepsilon \rightarrow 0$ and $\lambda \rightarrow \lambda^*$ by (5.3), provided α satisfies (5.2).

Thus, if α satisfies (5.2) and $2(\beta+s-\alpha) < n$ we deduce that

$$(6.13) \quad |I_\rho| \leq C$$

with C independent of $\varepsilon > 0$ and $\lambda \in [0, \lambda^*)$.

By Lemma 5.4 we have $1 - \beta C_{n,s} A_{n,s,\beta} > 0$. Therefore, from (6.11), (6.12) and (6.13), and using a uniform bound for u_λ in $B_1 \setminus B_{1/2}$ we deduce (6.6).

Step 2. Conclusion.

(a) Assume first that $n < 2(s+2 + \sqrt{2(s+1)})$. Then, $n/2 - s < 1 + \sqrt{n-1}$ and we can choose α satisfying $n/2 - s < \alpha < 1 + \sqrt{n-1}$. Thus, $n - 2s < n/2 + \alpha - s$ and we may choose $\beta = n - 2s$ in (6.6), which implies that $\int_{B_1} f(u_\lambda) \rho^{-n+2s} dx \leq C$ with a constant independent of λ . By (6.1) we have

$$u_\lambda(0) \leq C_n \int_{B_1} \rho^{-n+2s} f(u_\lambda(\rho)) dx \leq C.$$

Since u_λ is radially decreasing, we conclude that u_λ is uniformly bounded in B_1 as $\lambda \rightarrow \lambda^*$.

(b) Now assume that $n \geq 2(s+2 + \sqrt{2(s+1)})$. Suppose $1 < \alpha < 1 + \sqrt{n-1}$, $\beta > 0$, and $2(\beta+s-\alpha) < n$. Then, using that $f' > 0$, that u_λ is radially decreasing, as well as the estimate (6.6), we have for $\rho \leq 1/2$

$$c \rho^{n-\beta} f(u_\lambda(\rho)) = f(u_\lambda(\rho)) \int_{B_{2\rho} \setminus B_\rho} |x|^{-\beta} dx \leq \int_{B_1} f(u_\lambda) |x|^{-\beta} dx \leq C$$

where $c > 0$. This yields

$$f(u_\lambda(\rho)) \leq C\rho^{\beta-n} \quad \text{for } 0 < \rho \leq 1$$

where C is independent of λ . Using (6.1), this implies that if additionally $\beta < n-2s$, then

$$u_\lambda(x) \leq \frac{C}{|x|^{n-\beta-2s}} \quad \text{for all } x \in B_1.$$

Since we have the restrictions $\beta < n/2 + \alpha - s$ and $\alpha < 1 + \sqrt{n-1}$, we see that for any $\mu < n/2 - s - 1 - \sqrt{n-1}$, there is C independent of λ such that

$$u_\lambda(x) \leq \frac{C}{|x|^\mu} \quad \text{for all } x \in B_1.$$

By letting $\lambda \rightarrow \lambda^*$ in the last expression we conclude the proof. \square

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